Reliability Analysis Based on Warranty Database

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Abstract

Two alternative methods, maximum likelihood and semi-parametric, are described for estimating product lifetimes from warranty claims data only. They consider two lifetime variables: exponential failure, which corresponds to random failure mode, and Weibull failure, which corresponds to wear-out failure mode. The methods use only usage-at-failure data including exponential failure data. Simulation demonstrated their applicability.

Keywords: Reliability, random failure, wear-out failure, usage time distribution, censoring distribution.

1 Introduction

Unlike lifetime data sourced from the laboratory, lifetime data sourced from the field reflect the effects of environmental conditions and usage on a product while it is in service. Information obtained from analyzing such data thus gives manufacturers a better idea of the true reliability of their products. A commonly used source of field data is a warranty database because such databases are automatically generated and updated at no additional cost from repair claims that manufacturers receive during a products warranty period. Furthermore, it is safe to assume that customers almost always make a warranty claim when a product fails, so no record means no failure during the warranty period. A class of problems can be identified in warranty databases [1]. They would contain the complete data about warranty failures, but there is no information on usage time of non-failure products during the warranty period. For example, warranty data cannot provide the accumulated mileage of automobiles that did not fail during the warranty period.

Analyzing lifetime reliability involves two important aspects: age-based (e.g., calendar time measured lifetime) and usage-based (e.g., lifetime measured in mileage for automobiles, copy volume for photocopier machines, etc). While there have been many investigations of age-based analysis using warranty data ([2] – [6]), we focused on usage-based analysis because it is more relevant for engineering purposes. Since the usage-measured lifetime distributions for products that fail during the warranty period differ from those for products that fail after the warranty period, estimation of lifetime distributions generally requires data from sources other than warranty database. This information is important to know the partial information of the censored usage time or usage time distribution. Such sources include customer surveys, follow-up studies, periodic inspections, recall data and so on ([7] – [13]). However, generating these sources can be costly and in some cases even impossible. The unavailability of usage time for the censored units (i.e., the units that do not fail during the warranty period) makes estimating the lifetime distribution difficult. This problem has been addressed by Philips and Sweeting [15] by considering one lifetime variable: exponential failure.

We have extended the method of Alam and Suzuki [14] to consider two lifetime variables: exponential failure, which corresponds to random failure mode, and Weibull failure, which corresponds to wear-out failure mode. We developed two estimation methods, maximum likelihood and semi-parametric, that use only usage-at-failure data including exponential failure data.

In this paper we discuss the properties of these methods that support their applicability. Without loss of generality, we use the automobile as an illustrative system. The rest of this paper is organized

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as follows. Section 2 presents the data structure and model settings. Section 3 describes the two proposed methods for estimating the lifetime distribution. Section 4 discusses the properties of the two methods and the simulation results obtained using them. Section 5 summarizes the key points of the paper and outlines possible future developments of this research.

2 Data and Model Setting

The failure mechanisms of automobiles depend on both time in service and mileage. The type of data shown in Table 1 is generally available to manufacturers for automobiles that failed during the warranty period: months in service (MIS — time elapsed from sales to failure), accumulated mileage-at-failure, failure mode, etc. Automobiles that fail during the warranty period as a group typically have a large variety of failure modes, each having its own mileage-to-failure distribution. This distribution generally differs from the mileage-to-failure distribution of automobiles that do not fail during the warranty period because some failure modes tend to occur when the mileage is high or low.

In this study, two failure modes are considered. One is an increasing-failure rate, wear-out failure mode, in which the number of failures increases with mileage. The other is a constant-failure rate, random failure mode, in which the failure rate stays the same throughout the lifetime.

Month of Sale	Sales amount	Months in Service (MIS), t				
		1	...	t	...	T
1	N_1	//	...	////	...	//
...		...	...	...	...	
s	N_s	///	...	///		
...		...	...			
S	N_S	/				
Total	N	r_1		r_t		r_T

Table 1. Information available from warranty database

We denote the failure modes as a and b, with a being the wear-out failure mode and b being the random one. Let $X(a)$ denote the lifetime random variable corresponding to failure mode a measured in terms of mileage and $X(b)$ be the lifetime random variable corresponding to mode b, and let Y_t be a random variable (censoring random variable) representing the total mileage accumulated by an automobile at t MIS; it is independent of $X(a)$ and $X(b)$. Further, let $X(a)$ have a pdf of $f_a(.)$ and a cdf of $F_a(.)$ and $X(b)$ have a pdf of $f_b(.)$ and a cdf of $F_b(.)$, and let the censoring random variable have a pdf of $g_t(.)$ and a cdf of $G_t(.)$.

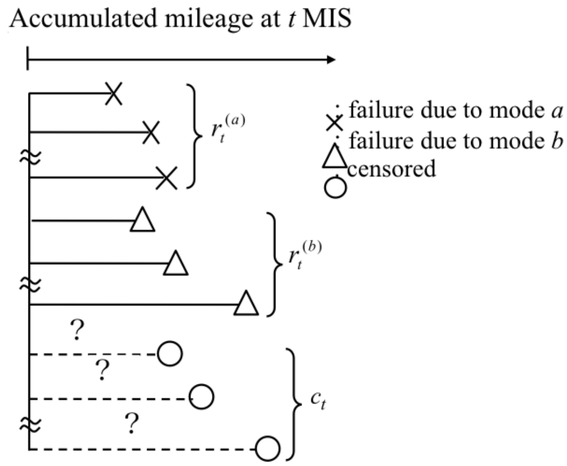

Figure 1. Usage time, i.e., mileage-based-data structure

The following competing risks model is often used for censoring problems in which all the observations, (U_{ti}, δ_{ti}), $i = 1, ..., n_t$, $t = 1, ..., T$, have been obtained:

$$U_{ti} = min\{X_i^{(a)}, X_i^{(b)}, Y_{ti}\} \text{ and}$$

$$\delta_{t,i} = \begin{cases} a \text{ if failed with mode } a \\ b \text{ if failed with mode } b \\ 0 \text{ if censored} \end{cases}$$

Several studies have been done on the censoring problem. Here, as shown in Table 1 and Figure 1, we are concerned with analyzing warranty data. Failure mileage is available if $min(X_i^{(a)}, X_i^{(b)}) \leq Y_{ti}$. Censored mileage for which $\delta_{t,i} = 0$ is not available from warranty claim records, so we can only use $Pr[min(X_i^{(a)}, X_i^{(b)}) > Y_{t,i}]$ for the likelihood function (LF). The problem is to find a way to estimate the model parameters on the basis of only the failure mileage.

Rai and Singh [10] demonstrated that a lognormal distribution provides a good fit for mileage accumulation data with a linear trend between mileage at t MIS, Y_t versus MIS t. These data are drawn from vehicle recall data and can be assumed to be a random sample with respect to mileage and calendar time. They showed that $Y_t \sim LN(\mu + \log t, \sigma^2)$. The same findings were obtained for the usage accumulation data for a particular type of photocopier machine. Figure 2 shows the usage accumulation data for a random sample of 255 photocopiers drawn from a population of 9000 that included both failed and non-failed machines. As shown in Figure 2, it is reasonable to assume a linear trend between copy volume, say A_t at t MIS, and MIS t. Figure 3 shows a histogram of the copy volume accumulation rate per MIS (CVAR) in logarithmic scale. As shown in Figure 3, a lognormal distribution also provides a good fit for the copy volume accumulation data. That is, $A_t = \alpha t$; random variable α follows a

Lognormal distribution with location parameter μ and scale parameter σ ($\log\alpha \sim N(\mu,\sigma^2)$).

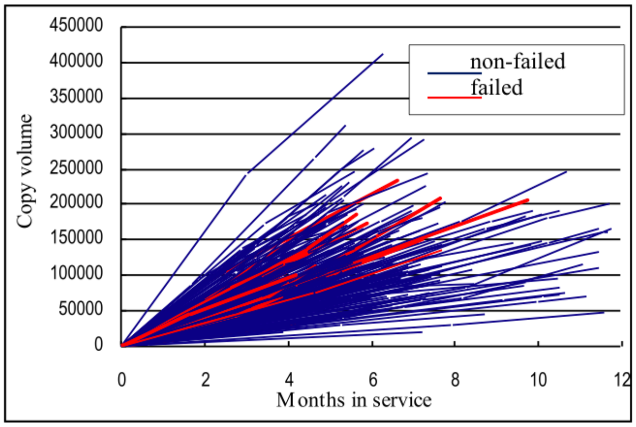

Figure 2. Copy volume vs. MIS of photocopier machines.

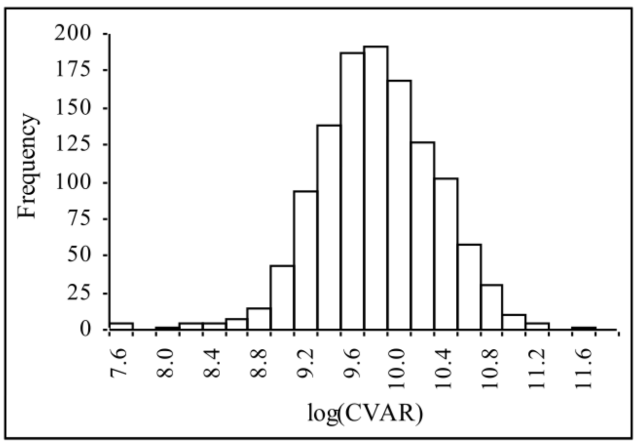

Figure 3. Histogram of copy volume accumulation rate per MIS for photocopier machines.

We use the Weibull distribution, which is extensively used for modeling mileage due to wear-out failure, and we assume that $X^{(a)}$ is the Weibull random variable with shape parameter m and scale parameter η, that is $X^{(a)} \sim Weibull(m,\eta)$. We assume that $X^{(b)}$ follows an Exponential distribution with mean $1/\lambda$, that is $X^{(b)} \sim Exp(\lambda)$.

3 Proposed Methods

3.1 Maximum likelihood (ML) method

Let Θ be the vector of parameters, and let $x_t^{(a)} = (x_{t,1}^{(a)},...,x_{t,r_t^{(a)}}^{(a)})'$ and $x_t^{(b)} = (x_{t,1}^{(b)},...,x_{t,r_t^{(b)}}^{(b)})'$ be the vectors of observed failure mileages for failure modes a and b, respectively, at t MIS ($t=1,2,...,T$). The likelihood function (LF) can then be obtained as

$$\begin{aligned} L(\Theta|\mathbf{x}) &= \prod_{t=1}^{T}\left[\prod_{i=1}^{r_t^{(a)}}\left(f_a(x_{ti}^{(a)})\bar{F}_b(x_{ti}^{(a)})\bar{G}_t(x_{ti}^{(a)})\right)\times\right. \\ &\times \prod_{i=1}^{r_t^{(b)}}\left(f_b(x_{ti}^{(b)})\bar{F}_b(x_{ti}^{(b)})\bar{G}_t(x_{ti}^{(b)})\right)\times \\ &\times \left.\left(\int_0^\infty g_t(\tau)\bar{F}_a(\tau)\bar{F}_b(\tau)\partial\tau\right)^{c_t}\right] \qquad (1) \end{aligned}$$

where $r_t^{(a)}$ and $r_t^{(b)}$ are the number of failures at t MIS for modes a and b, respectively, and c_t is the number of censored units at t MIS. Now we have:

$$\begin{aligned} f_a(w) &= (m/\eta)(w/\eta)^{m-1}e^{-(w/\eta)^m} \\ \bar{F}_a(w) &= e^{-(w/\eta)^m} \\ f_b(w) &= \lambda e^{-\lambda w} \\ F_b(w) &= e^{-\lambda w} \\ g_t(w) &= \frac{1}{\sqrt{2\pi}\sigma w}e^{-(\log w-\mu-\log t)^2/2\sigma^2} \\ \bar{G}_t(w) &= \bar{\Phi}\left((\log w-\mu-\log t)/\sigma\right) \end{aligned}$$

where $\bar{\Phi}$ is the survival function of the Standard Normal distribution. We define and transform:

$$\begin{aligned} J_t(\Theta) &= \int_0^\infty g_t(\tau)\bar{F}_a(\tau)\bar{F}_b(\tau)\partial\tau \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m}e^{-\gamma(w)^2}dw \qquad (2) \end{aligned}$$

where $\gamma(w) = \frac{(w-\mu-\log t)^2}{2\sigma^2}$. Using this definition in (1) and some simplification, we obtain the log-likelihood function:

$$\begin{aligned} \log L &= \sum_{t=1}^{T}\sum_{i=1}^{r_t^{(a)}}\left[(m-1)\log x_{ti}^{(a)} - m\log\eta + \right. \\ &+ \left.\log m - \left(x_{ti}^{(a)}/\eta\right)^m - \lambda x_{ti}^{(a)} + \log\overline{\Phi}(z_{ti}^{(a)})\right] + \\ &+ \sum_{t=1}^{T}\sum_{i=1}^{r_t^{(b)}}\left[\log\lambda - \lambda x_{ti}^{(b)} - \left(x_{ti}^{(b)}/\eta\right)^m + \right. \\ &+ \left.\log\overline{\Phi}(z_{ti}^{(b)})\right] \\ &+ \sum_{t=1}^{T} c_t\log J_t(\Theta) \qquad (3) \end{aligned}$$

where $z_{ti}^{(a),(b)} = \left(\log x_{ti}^{(a),(b)} - \mu - \log t\right)/\sigma$.

As shown by Alam and Suzuki [14], by using the Gauss-Hermite quadrature method ([17], for example) to compute the improper integral, $J_t = J_t(\Theta)$, in (2) and other J_t-type improper integrals (all the first and second order derivatives of J_t w.r.t. m, η, λ, μ, and σ are in the form of (2), see Appendix) in the Gradients and Hessians, we can maximize (3) to obtain the ML estimators (MLEs) of parameters m, η, λ, μ, and σ ($\hat{m}, \hat{\eta}, \hat{\lambda}, \hat{\mu}$, and $\hat{\sigma}$) simultaneously. Large sample theory can be used to make statistical inferences (hypothesis tests and confidence intervals) on model parameters, which states that, under certain regularity conditions, the estimated parameter vector $(\hat{m}, \hat{\eta}, \hat{\lambda}, \hat{\mu}, \hat{\sigma})$ asymptotically follows a Multivariate normal distribution with mean vector $(m, \eta, \lambda, \mu, \sigma)$ and a Variance-covariance matrix equal to the inverse of the Fisher information matrix. By using the observed data to estimate the Fisher information matrix, we can make statistical inferences.

3.2 Semi-parametric Method

The ML estimation method discussed above could be of limited use for a higher number of failure modes due to the complex form of the LF. Our second proposed method can be used to estimate five parameters by following a semi-parametric procedure and can be extended to a higher number of failure modes. There are four steps in this procedure.

3.2.1 Step One: ML estimation of λ, μ, and σ ignoring failure due to wear-out

We use the method of Alam and Suzuki [14] to estimate λ, μ, and σ, ignoring the failures due to wear-out. Then, the failures for random failure mode constitute the data set of interest, and we formulate the LF as [14]):

$$L(\Theta) = \prod_{t=1}^{T} \left[\prod_{i=1}^{r_t^{(b)}} \left(f_b(x_{ti}) \bar{G}_t(x_{ti}) \right) \times \times \left(\int_0^\infty g_t(\tau) \bar{F}_b(\tau) \partial\tau \right)^{c_t} \right] \quad (4)$$

which is same as the LF of Alam and Suzuki [14] with three unknown parameters. The MLEs of λ, μ, and σ ($\hat{\lambda}, \hat{\mu}$, and $\hat{\sigma}$) are obtained by using the ML method of Alam and Suzuki [14]

3.2.2 Step Two: Estimation of m and η

Once $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\sigma}$ are estimated in Step One, parameters m and η in (3) are estimated using the nonparametric MLE of Wang and Suzuki [16] and a probability plotting procedure. For the two failure modes considered here, the nonparametric MLE of $F_a(x)$ is

$$\hat{F}_a(x) = \sum_{x_k \le x} \frac{d_a(x_k)}{\sum_{s=1}^{S} N_s \bar{G}_{S-s+1}(x_k) \bar{F}_b(x_k)} \quad (5)$$

where $d_a(x_k)$ denotes the number of failures at support point x_k with failure mode a $(x_k < x_k+1)$. Using estimates $(\hat{\lambda}, \hat{\mu}, \hat{\sigma})$ we easily compute $F_b(.)$ and $G_t(.)$ in (5). We then obtain the nonparametric MLE $\hat{F}_a(x)$. By applying the Weibull probability plot and using $\hat{F}_a(x)$ we obtain the estimates of m and η ($\hat{m}$ and $\hat{\eta}$).

3.2.3 Step Three: Re-estimation of λ

Although all five parameters have now been estimated, there could be bias in the estimations of λ, μ, and σ due to the treatment to the mileage data of the Weibull failures in Step One. This could lead to biased estimates of m and η in Step Two.

Analysis of the simulation results revealed that, although the estimators of μ and σ proved robust, there were cases in which the estimation of λ was biased. Therefore, in Step Three, λ is re-estimated using the failure data of both the studied failure modes and the estimates other than $\hat{\lambda}$ obtained in Steps One and Two.

In this step, we consider the log-LF (3) as a function of λ alone, taking $\hat{\mu}$ and $\hat{\sigma}$ from Step One and $\hat{m}$ and $\hat{\eta}$ from Step Two. Following the ML approach, we then obtain an improved estimator of λ from (3)

3.2.4 Step Four: Re-estimation of m and η

Using the updated $\hat{\lambda}$ obtained in Step Three and the $\hat{\mu}$ and $\hat{\sigma}$ obtained in Step One, we reiterate Step Two to improve $\hat{m}$ and $\hat{\eta}$.

4 Properties of Proposed Methods and Simulation Results

4.1 Maximum likelihood method

A brief simulation was run to investigate the properties of the estimators obtained with the Maximum likelihood method. Data were generated assuming that $X(a) \sim Weibull(m, \eta)$, $X(b) \sim Exp(\lambda)$ and $Y_t \sim LN(\mu + \log t, \sigma^2)$ for five sets of true parameter values (third column in Table 2). We set μ to 7.3 and σ to 0.7 on the basis of actual data [10]. The other three parameters were set to make the simulation realistic. The sales amount for each month, $N_s (s = 1, ..., 12)$, was set to 500, so the total number of units (N) was 6000 (= 12 × 500). The observation period (M) and warranty period (W) were set to 12 months, so that MIS, t = 1,...,min (W, M) = 12. The five parameters were iteratively estimated using the log-LF (3) and the associated Gradients and Hessians (see Appendix). All the improper integrals involved in (3), Gradients, and Hessians were computed using the Gauss-Hermite quadrature method taking the integration order q to be

25. The MLEs of parameters m, μ, λ, μ, and σ were computed iteratively. The asymptotic variances of the estimates were calculated from the inverse of the observed Fisher information matrix. The whole process was repeated 1000 times, and the averages of the estimates and asymptotic variances were calculated. The percentage of biases was also calculated. The results are presented in Table 2.

We repeated the proceeding simulation with a monthly sales amount of 2000 to determine the effect of using a fairly larger sample size. As shown in Table 2, the biases as well as the asymptotic variances of the estimates were moderately lower, therefore the estimators appear consistent.

Table 2. True values, average of estimates, estimated asymptotic variances, percentage of bias, and average number of failures for 1000 runs and five data sets with $N_s = 500$ and $N_s = 2000$.

Data set	Parameter	True value	$N_s = 500$					$N_s = 2000$				
			Average of estimates	Average asymptotic variance	Bias (%)	$r(a)$ *	$r(b)$ **	Average of estimates	Average asymptotic variance	Bias (%)	$r(a)$ *	$r(b)$ **
1	m	2.00E+00	2.05E+00	7.11E-02	2.68	18 (0.29%)	147 (2.45%)	2.00E+00	1.86E-02	0.06	78 (0.33%)	580 (2.41%)
	η	3.00E+05	3.26E+05	4.02E+10	8.63			2.97E+05	7.69E+09	0.92		
	λ	2.00E-06	1.91E-06	1.89E-12	4.38			1.96E-06	9.50E-13	2.18		
	μ	7.30E+00	7.32E+00	8.60E-01	0.23			7.30E+00	4.64E-01	0.06		
	σ	7.00E-01	7.14E-01	1.51E-01	1.98			7.10E-01	8.07E-02	1.37		
2	m	2.50E+00	2.67E+00	4.50E-02	6.72	41 (0.69%)	46 (0.76%)	2.61E+00	1.07E-02	4.27	173 (0.72%)	178 (0.74%)
	η	1.40E+05	1.51E+05	1.05E+09	7.77			1.44E+05	2.59E+08	2.97		
	λ	6.00E-07	5.45E-07	6.01E-14	9.20			5.76E-07	1.85E-14	4.04		
	μ	7.30E+00	7.46E+00	2.20E-01	2.20			7.35E+00	7.58E-02	0.75		
	σ	7.00E-01	7.46E-01	3.25E-02	6.63			7.21E-01	1.10E-02	2.98		
3	m	3.00E+00	3.08E+00	6.65E-02	2.75	47 (0.78%)	60 (1.00%)	3.05E+00	2.04E-02	1.58	195 (0.81%)	234 (0.98%)
	η	1.10E+05	1.21E+05	6.53E+08	9.68			1.13E+05	3.24E+08	2.35		
	λ	8.00E-07	7.67E-07	1.62E-13	4.18			8.07E-07	1.39E-13	0.85		
	μ	7.30E+00	7.37E+00	3.74E-01	1.02			7.27E+00	3.16E-01	0.46		
	σ	7.00E-01	7.51E-01	5.98E-02	7.30			7.15E-01	5.18E-02	2.08		
4	m	1.50E+00	1.57E+00	1.65E-02	4.87	28 (0.46%)	34 (0.56%)	1.51E+00	4.40E-03	0.37	117 (0.49%)	133 (0.56%)
	η	5.00E+05	5.44E+05	1.25E+12	8.77			4.95E+05	1.87E+10	0.99		
	λ	4.50E-07	4.10E-07	3.35E-14	8.78			4.35E-07	1.20E-14	3.43		
	μ	7.30E+00	7.44E+00	3.28E-01	1.97			7.37E+00	1.09E-01	0.98		
	σ	7.00E-01	6.79E-01	6.10E-02	3.03			7.02E-01	1.73E-02	0.23		
5	m	2.00E+00	2.14E+00	2.32E-02	7.10	52 (0.86%)	45 (0.76%)	2.06E+00	5.55E-03	2.81	212 (0.88%)	179 (0.74%)
	η	1.75E+05	1.89E+05	1.99E+09	8.25			1.70E+05	3.17E+08	2.95		
	λ	6.00E-07	5.53E-07	8.65E-14	7.86			5.88E-07	1.28E-14	2.02		
	μ	7.30E+00	7.44E+00	3.16E-01	1.94			7.36E+00	5.06E-02	0.87		
	σ	7.00E-01	7.35E-01	4.90E-02	5.03			7.13E-01	7.77E-03	1.86		

*$r(a)$ denotes (average) number of failures for mode a.
**$r(b)$ denotes (average) number of failures for mode b.

4.2 Semi-parametric method

A simple simulation was also run to investigate the properties of the estimators obtained with the semi-parametric method. To enable us to compare the performance of this method with that of the ML method, we used the same parameter settings. The results are presented in Table 3.

As observed from Tables 2 and 3, the semi-parametric method had higher biases for the smaller sample size. As observed from Table 3, all the estimators with the semi-parametric method appear consistent between the two sample sizes.

5 Conclusions and Future Work

Failure mileage data for a random and a wear-out failure modes were used to estimate the lifetime distributions as well as censoring distribution. Simulation showed that, if one of the lifetime random variables follows an Exponential distribution, the other follows a Weibull distribution and the censoring time - a Log-normal distribution, warranty database information is sufficient to estimate the parameters of the three distributions.

Both of the proposed estimation methods showed consistent in the simulations. Consistency and other properties of the estimators are to be theoretically investigated in future work. The Maximum likelihood method enables statistical inferences but could be of limited use for more than two failure modes. The semi-parametric method produces estimates as good as those of the ML method and can be used for more than two failure modes. Three and more failure modes extension of the studied model is also due to be done in further work.

The bottom line is that both methods produce satisfactory results.

Table 3. True values, average of estimates, sampling variances, percentage of bias, and average number of failures for 1000 runs and five data sets with $N_s = 500$ and $N_s = 2000$.

Data set	Parameter	True Value	$N_s = 500$					$N_s = 2000$				
			Average of estimates	Sampling variance	Bias (%)	$r^{(a)}$ *	$r^{(b)}$ **	Average of estimates	Sampling variance	Bias (%)	$r^{(a)}$ *	$r^{(b)}$ **
1	m	2.00E+00	1.97E+00	1.11E-01	1.66	18 (0.29%)	147 (2.45%)	1.98E+00	2.65E-02	0.77	78 (0.33%)	580 (2.41%)
	η	3.00E+05	3.58E+00	1.10E+11	19.2			3.07E+05	4.94E+06	2.23		
	λ	2.00E-06	2.03E-06	4.76E-14	1.51			2.02E-06	1.12E-14	0.85		
	μ	7.30E+00	7.29E+00	6.40E-03	0.11			7.29E+00	1.90E-03	0.09		
	σ	7.00E-01	6.94E-01	1.62E-03	0.87			6.94E-01	4.16E-04	0.83		
2	m	2.50E+00	2.44E+00	7.67E-02	2.51	41 (0.69%)	46 (0.76%)	2.50E+00	2.11E-02	0.19	173 (0.72%)	178 (0.74%)
	η	1.40E+05	1.50E+05	1.32E+09	6.80			1.41E+05	2.46E+08	0.48		
	λ	6.00E-07	6.08E-07	1.36E-14	1.30			6.02E-07	3.28E-15	0.38		
	μ	7.30E+00	7.30E+00	2.43E-02	0.02			7.30E+00	5.80E-03	0.01		
	σ	7.00E-01	6.92E-01	5.78E-03	1.10			6.89E-01	1.38E-03	1.50		
3	m	3.00E+00	2.91E+00	8.73E-02	3.01	47 (0.78%)	60 (1.00%)	3.00E+00	2.60E-02	0.06	195 (0.81%)	234 (0.98%)
	η	1.10E+05	1.16E+05	3.34E+08	5.13			1.10E+05	6.80E+07	0.02		
	λ	8.00E-07	8.11E-07	1.83E-14	1.39			8.08E-07	4.56E-15	1.04		
	μ	7.30E+00	7.30E+00	1.77E-02	0.04			7.30E+00	4.39E-03	0.03		
	σ	7.00E-01	6.89E-01	4.41E-03	1.58			6.86E-01	1.14E-03	2.00		
4	m	1.50E+00	1.47E+00	6.09E-02	1.73	28 (0.46%)	34 (0.56%)	1.48E+00	1.48E-02	1.08	117 (0.49%)	133 (0.56%)
	η	5.00E+05	7.21E+05	9.91E+11	44.27			5.42E+05	3.11E+10	8.40		
	λ	4.50E-07	4.58E-07	1.02E-14	1.75			4.51E-07	2.57E-15	0.15		
	μ	7.30E+00	7.30E+00	3.23E-02	0.05			7.30E+00	8.02E-03	0.01		
	σ	7.00E-01	6.96E-01	7.91E-03	0.50			6.97E-01	2.02E-03	0.45		
5	m	2.00E+00	1.96E+00	4.56E-02	1.84	52 (0.86%)	45 (0.76%)	1.99E+00	1.19E-02	0.44	212 (0.88%)	179 (0.74%)
	η	1.75E+05	1.89E+05	3.30E+09	7.98			1.77E+05	5.69E+08	1.15		
	λ	6.00E-07	6.09E-07	1.37E-14	1.46			6.05E-07	3.39E-15	0.88		
	μ	7.30E+00	7.30E+00	2.43E-02	0.04			7.30E+00	6.15E-03	0.02		
	σ	7.00E-01	6.93E-01	5.76E-03	0.94			6.91E-01	1.60E-03	1.27		

*$r(a)$ denotes (average) number of failures for mode a.
**$r(b)$ denotes (average) number of failures for mode b.

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All computer simulations in this research were performed using R - R Development Core Team (2007). R: A language and environment for statistical computing. URL http://www.R-project.org.

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7 Appendix

The first and second order derivatives w.r.t. parameters λ, μ, σ, m and η of the improper integral in (2) are given below. With these the Gradient and Hessian derivation of the LF function (1) becomes trivial.

$$\frac{\partial J_t}{\partial \lambda} = \frac{-1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{w-\lambda e^w - (e^w/\eta)^m - \frac{(w-\mu-\log t)^2}{2\sigma^2}} dw \tag{6}$$

$$\frac{\partial J_t}{\partial \mu} = \frac{1}{\sqrt{2\pi}\sigma^3} \int_{-\infty}^{\infty} e^{-\lambda e^w - (e^w/\eta)^m - \frac{(w-\mu-\log t)^2}{2\sigma^2}} (w-\mu-\log t) dw \tag{7}$$

$$\begin{aligned} \frac{\partial J_t}{\partial \sigma} &= \frac{-1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} e^{-\lambda e^w - (e^w/\eta)^m - \frac{(w-\mu-\log t)^2}{2\sigma^2}} dw + \\ &+ \frac{1}{\sqrt{2\pi}\sigma^4} \int_{-\infty}^{\infty} e^{-\lambda e^w - (e^w/\eta)^m - \frac{(w-\mu-\log t)^2}{2\sigma^2}} (w-\mu-\log t)^2 dw \end{aligned} \tag{8}$$

$$\frac{\partial J_t}{\partial m} = \frac{-1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\lambda e^w - (e^w/\eta)^m - \frac{(w-\mu-\log t)^2}{2\sigma^2}} \log(e^w/\eta)\,(e^w/\eta)^m\, dw \tag{9}$$

$$\frac{\partial J_t}{\partial \eta} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\lambda e^w - (e^w/\eta)^m - \frac{(w-\mu-\log t)^2}{2\sigma^2}} (m/\eta)\,(e^w/\eta)^m\, dw \tag{10}$$

$$\frac{\partial^2 J_t}{\partial \lambda^2} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{2w-\lambda e^w - (e^w/\eta)^m - \frac{(w-\mu-\log t)^2}{2\sigma^2}} dw \tag{11}$$

$$\frac{\partial^2 J_t}{\partial \mu^2} = \frac{1}{\sqrt{2\pi}\sigma^5} \int_{-\infty}^{\infty} e^{-\lambda e^w - (e^w/\eta)^m - \frac{(w-\mu-\log t)^2}{2\sigma^2}} (w-\mu-\log t)^2 dw -$$

$$
\begin{aligned}
& - \quad \frac{1}{\sqrt{2\pi}\sigma^3}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}dw && (12)\\
\frac{\partial^2 J_t}{\partial \sigma^2} &= \frac{2}{\sqrt{2\pi}\sigma^3}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}dw - \\
& - \quad \frac{5}{\sqrt{2\pi}\sigma^5}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t)^2 dw + \\
& + \quad \frac{1}{\sqrt{2\pi}\sigma^7}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t)^4 dw && (13)\\
\frac{\partial^2 J_t}{\partial m^2} &= \frac{-1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}\left(\log(e^w/\eta)\,(e^w/\eta)^m\right)^2 dw - \\
& - \quad \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}\left(\log(e^w/\eta)\right)^2 (e^w/\eta)^m\, dw && (14)\\
\frac{\partial^2 J_t}{\partial \eta^2} &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}\left((m/\eta)\,(e^w/\eta)^m\right)^2 dw - \\
& - \quad \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}\left((m/\eta)^2\,(e^w/\eta)^m + (m/\eta)^2\,(e^w/\eta)^m\right) dw && (15)\\
\frac{\partial^2 J_t}{\partial m \partial \eta} &= \frac{-1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(e^w/\eta)^{2m}\,(m/\eta)\log e^w/\eta + \\
& + \quad \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(e^w/\eta)^m\,(1/\eta-(m/\eta)\log e^w/\eta)\, dw && (16)\\
\frac{\partial^2 J_t}{\partial \lambda \partial m} &= \frac{-1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{w-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}\left(-\log(e^w/\eta)\,(e^w/\eta)^m\right) dw && (17)\\
\frac{\partial^2 J_t}{\partial \lambda \partial \eta} &= \frac{-1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{w-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}\left((m/\eta)\,(e^w/\eta)^m\right) dw && (18)\\
\frac{\partial^2 J_t}{\partial \sigma \partial m} &= \frac{-1}{\sqrt{2\pi}\sigma^4}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t)^2\left(\log(e^w/\eta)\,(e^w/\eta)^m\right) dw + \\
& + \quad \frac{1}{\sqrt{2\pi}\sigma^2}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}\left(\log(e^w/\eta)\,(e^w/\eta)^m\right) dw && (19)\\
\frac{\partial^2 J_t}{\partial \sigma \partial \eta} &= \frac{1}{\sqrt{2\pi}\sigma^4}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t)^2\left((m/\eta)\,(e^w/\eta)^m\right) dw - \\
& - \quad \frac{1}{\sqrt{2\pi}\sigma^2}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(m/\eta)\,(e^w/\eta)^m\, dw && (20)\\
\frac{\partial^2 J_t}{\partial \sigma \partial \lambda} &= \frac{1}{\sqrt{2\pi}\sigma^2}\int_{-\infty}^{\infty} e^{w-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}dw - \\
& - \quad \frac{1}{\sqrt{2\pi}\sigma^4}\int_{-\infty}^{\infty} e^{w-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t)^2 dw && (21)\\
\frac{\partial^2 J_t}{\partial \mu \partial m} &= \frac{1}{\sqrt{2\pi}\sigma^3}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t)\left(-\log(e^w/\eta)\,(e^w/\eta)^m\right) dw && (22)\\
\frac{\partial^2 J_t}{\partial \mu \partial \eta} &= \frac{1}{\sqrt{2\pi}\sigma^3}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t)\left((m/\eta)\,(e^w/\eta)^m\right) dw && (23)\\
\frac{\partial^2 J_t}{\partial \mu \partial \lambda} &= \frac{-1}{\sqrt{2\pi}\sigma^3}\int_{-\infty}^{\infty} e^{w-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t) dw && (24)\\
\frac{\partial^2 J_t}{\partial \mu \partial \sigma} &= \frac{-3}{\sqrt{2\pi}\sigma^4}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t) dw + \\
& + \quad \frac{1}{\sqrt{2\pi}\sigma^6}\int_{-\infty}^{\infty} e^{-\lambda e^w-(e^w/\eta)^m-\frac{(w-\mu-\log t)^2}{2\sigma^2}}(w-\mu-\log t)^3 dw && (25)
\end{aligned}
$$